Gravitational Entropy from Gravitational Shear

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Chamblin and Erlich, hep-th/0301130

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Among the many deep questions Andrew pondered about the nature of gravity is this one:

Can the entropy of gravitational backgrounds be understood without quantum gravity?

Andrew and I spent some time thinking about this in the context of covariant entropy bounds, which led to some promising speculations that I will tell you about.

Gravitational Entropy

The concept of gravitational entropy first arises in the context of Black Holes.

Second law of thermodynamics in the presence of black holes $\rightarrow$ black holes have an associated entropy (Bekenstein and Hawking).

Study QFT in Schwarzschild geometry.

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 \, d\Omega^2.$$

Near the horizon this looks like Rindler space:

$$ds^2 = \frac{1}{16G^2M^2} \rho^2 \, dt^2 - d\rho^2 - G^2M^2 \, d\Omega^2.$$

Do QFT in Rindler space:

• Break up fields into portion interior/exterior to horizon

• Construct wave functional for quantum state

$$\Psi[\phi_0] \equiv \int [d\phi]\, e^{-S}.$$

Path integral over future half space $x^0 > 0$ w/ $\phi = \phi_0$ at $x^0 = 0$.

• Compute density functional

$$R \equiv \int [d\phi_{\text{in}}]\, \Psi^*[\phi_{\text{in}}, \phi_{\text{out}}]\Psi[\phi_{\text{in}}, \phi_{\text{out}}].$$

Result:
$R = \exp[-H/T]$
H=Hamiltonian conjugate to Rindler time.
$T = \frac{1}{2\pi}$.

This is a thermal density functional with Rindler temperature T.

• Transform to appropriate coordinates:

Near horizon:
proper temperature $T(\rho) = \frac{T}{\rho} = \frac{1}{2\pi\rho}$

Distant observer:
Hawking temperature $T_{\mathsf{H}} = \frac{1}{8\pi MG}$.

• First law of thermodynamics:

$$dM = T\,dS$$

$$\to S = 4\pi M^2 G = \frac{A_{\text{horizon}}}{4G}$$

• Supposed to be true for all spacetimes with horizons, not just Schwarzschild black hole.

• Can't do any better than making a black hole: Expect a statement like $S \leq \frac{A}{4G}$ to be true for any surface with area A.

Why this is strange

• Consider a system of spin-1/2 particles on lattice w/ spacing a, volume V.

#States = $2^{V/a^3}$ Exponential in Volume.

Similarly for QFT.

Black Holes $\rightarrow$ *Holography*

• Consider free massless fields.

Entropy in shell $(\rho, \rho + d\rho)$ given by

$$\frac{dS}{d\rho\, d^2x} \sim T^3.$$

Use near horizon result $T = 1/2\pi\rho \rightarrow$

$$S \sim A \int \frac{d\rho}{\rho^3}.$$

• Must break down at small ρ.

• Finiteness of Black Hole entropy $\rightarrow$ QFT overstates #degrees of freedom.

Other oddities with area scaling

• Volumes add. Area's don't. Consider two boxes touching at one face. What does the entropy bound say regarding the mutual face?

• What if there are no mutual faces? Consider a distribution of closely packed spheres. What does the entropy bound say about an enveloping sphere which has less area than the sum of the smaller areas?

There are many ways to obtain $S_{BH} \propto A$.

There are fewer ways to derive $S_{BH} = A/4G$.

One way: Consider a thin spherical shell of energy-carrying stuff at infinity.

- There is a local acceleration temperature at the shell, T_{acc}.
- Bring the shell in from infinity while keeping it in thermodynamical and mechanical equilibrium.
- In order to do this entropy must be added to the shell. Otherwise it would heat up too quickly.
- Then, if the equation of state is not too singular,

$$S = A/4G$$

when the shell reaches its own Schwarzschild radius.

(Pretorius, Vollick and Israel, gr-qc/9712085)

How can we make these ideas precise? And what might we learn about gravity?

- What do we mean by the area surrounding a volume?
- What do we mean by the volume inside an area?

These are not covariant concepts. The confusion arises from thinking about spacelike boundaries of volumes.

So, consider null surfaces → The Bousso Bound.

A few facts from GR:

• Geodesics are bent in the presence of stress tensor or Weyl tensor.

• If $k^\mu T_{\mu\nu} k^\nu > 0$ then null geodesics are focussed by both of these sources.

• Raychaudhuri eqn. describes this:

$$-\frac{d\theta}{d\lambda} = \frac{1}{2}\theta^2 + k^\mu T_{\mu\nu} k^\nu + \sigma_{\mu\nu}^2 \ (> 0)$$

θ is called the expansion. It describes how a congruence of null rays expands.

$\sigma_{\mu\nu}$ is called the shear. It describes how a congruence of null rays is sheared.

• The change in area of a null congruence is given by the area decrease factor,

$$A(\lambda) = \exp \int_0^\lambda d\tilde{\lambda}\, \theta(\tilde{\lambda}).$$

• At a caustic, where neighboring lightrays meet, $\theta \to -\infty$.

The Bousso Bound

• Take an arbitrary spacelike two-surface B.

• Construct a congruence of ingoing null rays from B. This is a congruence with $\theta \leq 0$ everywhere on B.

• Construct the light sheet Γ orthogonal to B. The lightsheet is terminated at caustics, boundaries or singularities.

• Then the entropy flux through Γ is bounded by

$$S_\Gamma \leq A_B/4G.$$

This is the Bousso bound.

Derivation of the Bousso Bound

Flanagan, Marolf and Wald, hep-th/9908070

Assume there is an entropy flux vector s^μ on the Bousso lightsheet Γ, such that

$$s^\mu k_\mu < (k^\mu T_{\mu\nu} k^\nu + \sigma_{\mu\nu}\sigma^{\mu\nu})(\lambda_\infty - \lambda)$$

(This is very much like the Bekenstein bound, $S < 2\pi E R$.)

Then

$$\int d\lambda\, d^2x\, s^\mu k_\mu < A_B/4G.$$

The derivation goes something like this:

• Consider each null ray independently and integrate over d^2x at the end.

• Use the Raychaudhuri eqn. to show that $\int d\lambda\, s^\mu k_\mu$ is bounded along any ingoing null ray.

In fact, a more general entropy bound can be derived. Cut off the lightsheet on a spacelike 2-surface with area A'. Then,

$$S < (A_B - A')/4G$$

Some more detail

We need to prove that

$$I \equiv \frac{1}{8}\int_0^1 d\lambda\,(1-\lambda)\,(\sigma^2 + kTk)\,\mathcal{A}(\lambda) < \frac{1}{4}(1-\mathcal{A}(1))$$

Define

$$G(\lambda) = \sqrt{A(\lambda)} = \sqrt{\text{exp}\left[\int_0^\lambda d\bar{\lambda}\,\theta(\bar{\lambda})\right]}$$

Then using Raychaudhuri we rewrite I as

$$\begin{aligned} I &= -\frac{1}{4}\int_0^1 d\lambda\,(1-\lambda)\,G''(\lambda)\,G(\lambda) \\ &= -\frac{1}{4}\int_0^1 d\lambda\,(1-\lambda)\,G''(\lambda) + \frac{1}{4}\int_0^1 d\lambda\,(1-\lambda)\,G''(\lambda)(1-G(\lambda)) \\ &= \frac{1}{4}\big[G(0) - G(1) + G'(0)\big] + \frac{1}{4}\int_0^1 d\lambda\,(1-\lambda)\,G''(\lambda)\,(1-G(\lambda)). \end{aligned}$$

But $G(0) = 1$ and $G(1) = \sqrt{\mathcal{A}(1)}$, so we can now write I as,

$$\begin{aligned} I &= \frac{1}{4}(1-\mathcal{A}(1)) + \frac{1}{4}G'(0) - \frac{1}{4}\left(\sqrt{\mathcal{A}(1)} - \mathcal{A}(1)\right) \\ &- \frac{1}{4}\int_0^1 d\lambda\,(1-\lambda)\,G''(\lambda)\,(G(\lambda)-1). \end{aligned}$$

$$I = \frac{1}{4}(1 - \mathcal{A}(1)) + \frac{1}{4}G'(0) - \frac{1}{4}\left(\sqrt{\mathcal{A}(1)} - \mathcal{A}(1)\right)$$
$$- \frac{1}{4}\int_0^1 d\lambda\,(1-\lambda)\,G''(\lambda)\,(G(\lambda)-1).$$

Each term except the term in red is negative. Hence,

$$I < \frac{1}{4}(1 - \mathcal{A}(1)).$$

Finally, integrating over d^2x on the two-surface B gives the generalized Bousso bound:

$$S < \frac{A_B - A'}{4G}$$

Let's assume that the assumptions which led to the Bousso bound are some deep facts about nature. Where might such conditions come from?

Null energy condition:

$$k^\mu T_{\mu\nu} k^\nu > 0$$

This says that $\rho + p$ should be positive.

This is satisfied by classical field configurations, can be violated by quantum fluctuations.

The FMW constraint:

$$s \cdot k < (kTk + \sigma^2)(1 - \lambda)$$

This says roughly that energy density and size are necessary in order to have nonvanishing entropy.

(You can put more stuff in a bigger box.)

But what is that σ^2 doing there?

Separation of quantum mechanics and gravity

Put back the $\hbar$'s and G's.

$$\hbar\, s \cdot k < \left(k \cdot T \cdot k + \frac{\sigma^2}{G} \right) (1 - \lambda)$$

The part of the bound involving $T_{\mu\nu}$ is a statement only about quantum mechanics. Gravity plays only a classical role in the Bousso mound.

On the other hand, the part of the bound involving σ^2 involves both $\hbar$ and G. If the FMW condition is a statement about the world, then this part of it seems like a statement about quantum gravity.

Let's say the FMW constraint is saturated. When is the Bousso bound saturated?

Consider the expression for I that appeared in the derivation.

$$\begin{aligned} I \; = \; & \frac{1}{4}(1-\mathcal{A}(1)) + \frac{1}{4}G'(0) - \frac{1}{4}\left(\sqrt{\mathcal{A}(1)} - \mathcal{A}(1)\right) \\ - \; & \frac{1}{4}\int_0^1 d\lambda\,(1-\lambda)\,G''(\lambda)\,(G(\lambda)-1). \end{aligned}$$

- $G'(0)$ vanishes only when the expansion θ vanishes on B. This can happen on a black hole horizon.

- $\mathcal{A}(1) = \sqrt{\mathcal{A}(1)}$ if $\mathcal{A}(1) = 0$ or 1.

$\mathcal{A}(1) = 0$ means the null ray is allowed to reach a caustic.

$\mathcal{A}(1) = 1$ means then null ray is stuck to the horizon or is immediately cut off.

- The remaining integral vanishes if $(kTk + \sigma^2)$ vanishes whenever $\mathcal{A}(\lambda) \neq 1$.

This means that only a thin shell of matter sitting at its Schwarzschild radius can saturate the bound. This is reminiscent of the thin shell brought in from infinity which gave the same result.

Let's calculate I for the thin shell.

Minkowski space inside shell; Schwarzschild outside.

Both metrics have the form

$$ds^2 = f(r)\,dt^2 - h(r)\,dr^2 - r^2\,d\Omega^2.$$

Killing vector $\partial_t \rightarrow$

$$\frac{dt}{d\lambda} = E\,g^{tt} = E/f$$

$ds^2 = 0 \rightarrow$

$$\frac{dr}{d\lambda} = \frac{E}{\sqrt{fh}}$$

Minkowski: $f = f_0 = (1 - 2GM/R)$, $h = 1$.
Schwarzschild: $f = (1 - 2GM/r)$, $h = 1/f$.

$r \in (R, 0)$ when $\lambda \in (\lambda_0, 1)$. Then

$$E = \frac{R\sqrt{f_0}}{(1 - \lambda_0)}.$$

Stress tensor has the form

$$T_\mu^\nu = S_\mu^\nu \, \frac{\delta(r - R)}{\sqrt{g_{rr}}}.$$

Israel junction condition determines $S_\mu^\nu \rightarrow$

$$S_t^t = \frac{2}{R} \Delta \left(\frac{1}{\sqrt{h}} \right).$$

That's all we need to calculate $I \rightarrow$

$$I = \frac{1}{8} \int_0^1 d\lambda \, kTk \, (1 - \lambda) = \frac{1}{4} \left(1 - \sqrt{f_0} \right).$$

The Bousso bound is saturated when the shell sits at the horizon.

Notice that the FMW condition leads to a bound of the form

$$S < \frac{A_{horizon}}{4G},$$

no matter how big A_B is.

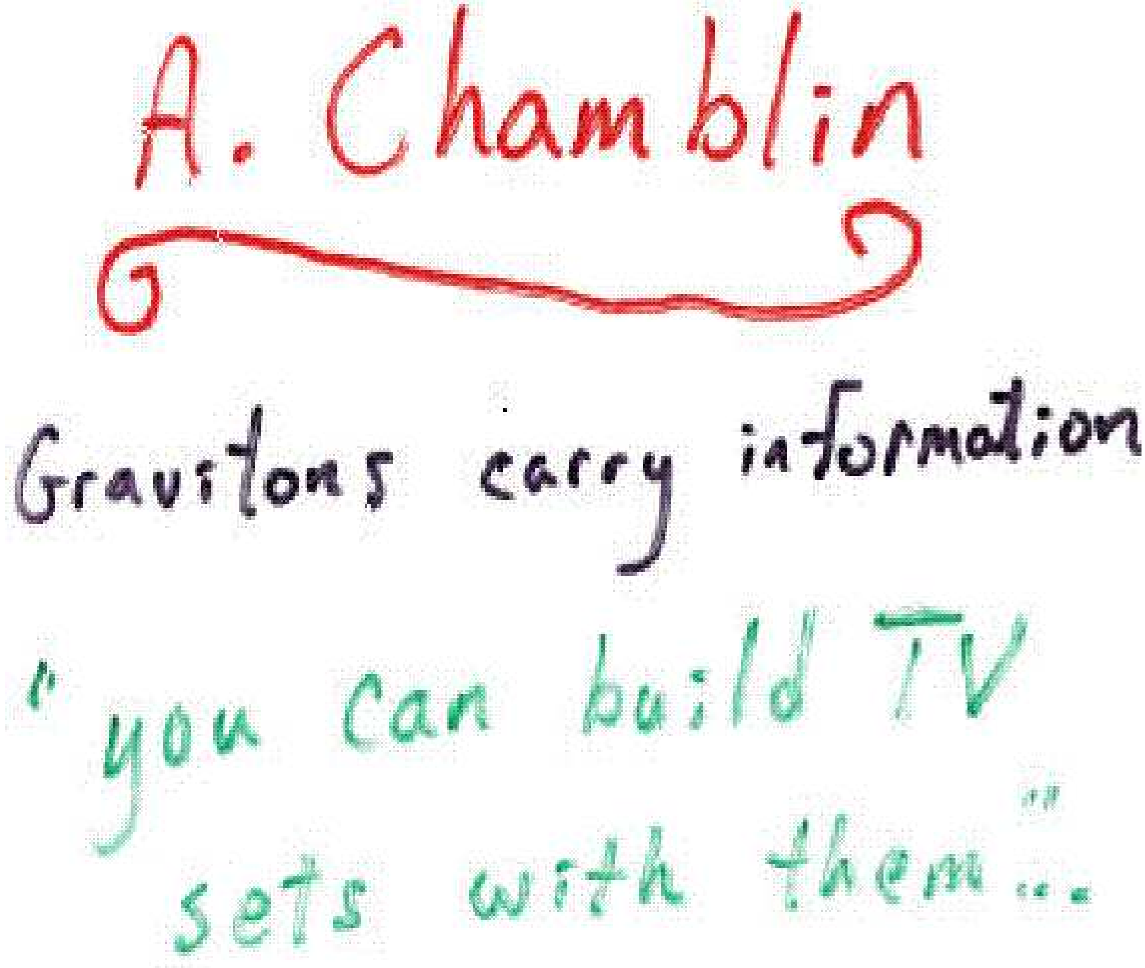
Gravitational Entropy

In trying to understand the concept of energy in GR, Penrose suggested that the shearing of null congruences might provide such a definition.

It's only a small leap to associate shear with gravitational entropy.

By gravitational entropy I mean any entropy in a weakly gravitating system that is not associated with the existence of $T_{\mu\nu}$.

For example, consider a Schwarzschild black hole. There is no stress tensor anywhere, but there is an associated entropy.



Andrew Chamblin
KITP String Cosmology Workshop, 11-13-2003

The FMW condition becomes

$$s \cdot k < \sigma^2 \, (1 - \lambda)$$

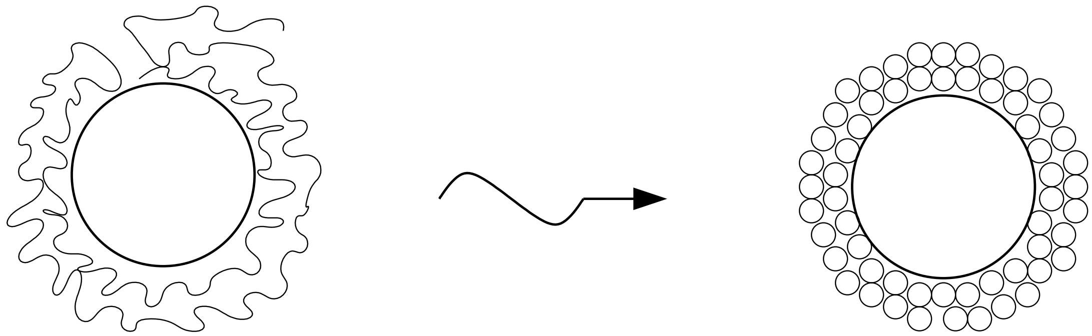
But a spherically symmetric lightsheet has no shear on it in a spherically symmetric spacetime. So for a Schwarzschild black hole the FMW condition implies a vanishing entropy.

Any sensible definition of black hole entropy would give $S = A/4G$. So either we say that black holes violate the FMW condition, or we ask a different question.

A different question: Which lightsheet should we use to describe gravitational entropy?

The answer: The more deformed the lightsheet, the more shear it sees. So perhaps a very deformed lightsheet close to the horizon will give the right answer.

Let's see if this is true...



Size of small lightsheets $\sim L$.

Integrate over lightsheets as close to horizon as $r \simeq 2M + \delta$

$$\sigma_{tr} \sim \nabla_{(t}k_{r)} \sim \frac{LM}{r^2}\frac{1}{(1-2M/r)}$$

Then, $S_{shell} \sim (\sigma_{tr})^2 \frac{r^2}{L^2} L^2$,

$$S \sim \int_{2M+\delta}^{\infty} \frac{dr}{L} S_{shell} = \frac{M^2 L}{\delta}.$$

If the regulators L and δ are both chosen to be around the Planck scale, then,

$$S \sim M^2 \sim A_{horizon}.$$

Conclusions

• A relation similar to the Bekenstein bound leads to a generalized area bound on the entropy of a weakly gravitating system.

• The Bekenstein-like bound appears to distinguish between quantum mechanics and gravity.

• At least in static spherically symmetric case, there is a stronger bound:

$$S < \frac{A_{horizon}}{4G}$$

where $A_{horizon}$ is the horizon area of a black hole with the ADM mass of the system.

• The contribution of the shear to the bound might be interpreted as due to gravitational entropy, and perhaps selects a preferred class of lightsheets hugging the horizon → Membrane paradigm?

• In the presence of matter, there may be a duality between the gravitational entropy of small lightsheets and the matter entropy of large lightsheets. Open-closed string duality?